Descent c-Wilf Equivalence

Quang T. Bach

Department of Mathematics
University of California, San Diego

Joint work with Professor Jeffrey Remmel

June 27, 2016
Let $S_n$ be the symmetric group and let $\sigma = \sigma_1 \ldots \sigma_n \in S_n$. Define

- $\text{Des}(\sigma) = \{i : \sigma_i > \sigma_{i+1}\}$ and $\text{des}(\sigma) = |\text{Des}(\sigma)|$,
- $\text{inv}(\sigma) = |\{(i,j) : 1 \leq i < j \leq n \text{ and } \sigma_i > \sigma_j\}|$,
- $\text{maj}(\sigma) = \sum_{i \in \text{Des}(\sigma)} i$.
- The reduction of $\sigma$, $\text{red}(\sigma)$, is the permutation found by replacing the $i^{th}$ smallest integer that appears in $\sigma$ by $i$.

For example, if $\sigma = 2 \ 7 \ 5 \ 4$, then $\text{red}(\sigma) = 1 \ 4 \ 3 \ 2$. 
Given $\tau = \tau_1 \cdots \tau_j \in S_j$ and $\sigma = \sigma_1 \cdots \sigma_n \in S_n$, we say that

- $\tau$ occurs in $\sigma$ if there exists $1 \leq i_1 < \cdots < i_j \leq n$ such that $\text{red}(\sigma_{i_1} \sigma_{i_2} \cdots \sigma_{i_j}) = \tau$,
- $\sigma$ avoids $\tau$ is there is no occurrence of $\tau$ in $\sigma$,
- there is a $\tau$-match starting in position $i$ in $\sigma$ if $\text{red}(\sigma_i \sigma_{i+1} \cdots \sigma_{i+j-1}) = \tau$.

Let $S_n(\tau)$ denote the set of permutations of $S_n$ which avoid $\tau$ and $N\mathcal{M}_n(\tau)$ denote the set of permutations of $S_n$ which have no $\tau$-matches.

Let $S_n(\tau) = |S_n(\tau)|$ and $N\mathcal{M}_n(\tau) = |N\mathcal{M}_n(\tau)|$. 
Let $S_n(\tau)$ denote the set of permutations of $S_n$ which avoid $\tau$ and $\text{NM}_n(\tau)$ denote the set of permutations of $S_n$ which have no $\tau$-matches. Let $S_n(\tau) = |S_n(\tau)|$ and $\text{NM}_n(\tau) = |\text{NM}_n(\tau)|$.

If $\alpha$ and $\beta$ are elements of $S_j$, then we say that

- $\alpha$ is Wilf-equivalent to $\beta$ if $S_n(\alpha) = S_n(\beta)$ for all $n \geq 1$ and
- $\alpha$ is c-Wilf-equivalent to $\beta$ if $\text{NM}_n(\alpha) = \text{NM}_n(\beta)$ for all $n \geq 1$. 
All patterns of length 3 are Wilf equivalent: There is only one Wilf equivalence class \{123, 321, 132, 312, 213, 231\} and

\[ S_n(132) = C_n = \frac{1}{n+1} \binom{2n}{n}. \]

For patterns of length 4, there are three Wilf equivalence classes:

<table>
<thead>
<tr>
<th>Pattern Class</th>
<th>Author</th>
</tr>
</thead>
<tbody>
<tr>
<td>{1234, 1243, 1432, 2134, 2143, 2341, 3142, 3214, 3412, 4123, 4312, 4321}</td>
<td>Bóna</td>
</tr>
<tr>
<td>{1342, 1423, 2314, 2413, 3142, 2431, 3124, 3241, 4132, 4213}</td>
<td>Gessel</td>
</tr>
<tr>
<td>{1324, 4231}</td>
<td>open</td>
</tr>
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</table>
Refinements of the Wilf Equivalent
For any permutation statistic $\text{stat}$ on permutations and any pair of permutations $\alpha$ and $\beta$ in $S_j$, we say that $\alpha$ is $\text{stat}$-Wilf equivalent to $\beta$ if for all $n \geq 1$

$$\sum_{\sigma \in S_n(\alpha)} \chi^{\text{stat}}(\sigma) = \sum_{\sigma \in S_n(\beta)} \chi^{\text{stat}}(\sigma).$$

We give examples for $\text{stat}$-Wilf equivalent in permutations of length 3 in the case $\text{stat} = \text{des}, \text{inv}, \text{maj}$, respectively.
des-Wilf Equivalent

\[ T(n, k) = \sum_{\sigma \in S_n(\tau)} x^{\text{des}(\sigma)} |_{x^k} \]

| Equiv. Class | Equiv. Class | T(n, k) = \sum_{\sigma \in S_n(\tau)} x^{\text{des}(\sigma)} |_{x^k} |
|--------------|--------------|-------------------------------------------------|
| 123          | 123          | A091156                                         |
| 132          | 132          |                                                 |
| 213          | 213          | Narayana numbers - A001263 (added by Zabrocki - 2004) |
| 231          | 231          |                                                 |
| 312          | 312          |                                                 |
| 321          | 321          | A091156 (added by Baxter - 2011)                 |
inv-Wilf Equivalent

| Equiv. Class | Equiv. Class | \( T(n, k) = \sum_{\sigma \in S_n(\tau)} x^{\text{inv}(\sigma)} |_{x^k} \) |
|--------------|--------------|--------------------------------------------------|
| 123          | 123          | Not in OEIS                                      |
| 132          | 132          | A129176                                          |
| 213          | 213          |                                                 |
| 231          | 231          | A227543                                          |
| 312          | 312          |                                                 |
| 321          | 321          | A140717                                          |
maj-Wilf Equivalent

\[
T(n, k) = \sum_{\sigma \in S_n(\tau)} x^{\text{maj}(\sigma)} |_{x^k}
\]

<table>
<thead>
<tr>
<th>Equiv. Class</th>
<th>Equiv. Class</th>
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</tr>
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<tbody>
<tr>
<td>123</td>
<td>123</td>
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</tr>
<tr>
<td>132</td>
<td>132</td>
<td>Not in OEIS</td>
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<tr>
<td>213</td>
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<td>231</td>
<td>213</td>
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<td>312</td>
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<td>321</td>
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</tbody>
</table>
Our Goal: study refinements of the c-Wilf equivalence relation.

For any permutation statistic $\text{stat}$ on permutations and any pair of permutations $\alpha$ and $\beta$ in $S_j$, we say that $\alpha$ is $\text{stat}$-c-Wilf equivalent to $\beta$ if for all $n \geq 1$

$$\sum_{\sigma \in N\mathcal{M}_n(\alpha)} x^\text{stat}(\sigma) = \sum_{\sigma \in N\mathcal{M}_n(\beta)} x^\text{stat}(\sigma).$$
Duane and Remmel showed that there is a large number of examples of $\alpha$ and $\beta$ which are $\text{inv-c-Wilf}$ equivalent when $\alpha$ and $\beta$ are minimal overlapping permutations.

We say that a permutation $\tau \in S_j$ where $j \geq 3$ is \textit{minimal overlapping} if in any permutation $\sigma = \sigma_1 \ldots \sigma_n$, any two $\tau$-matches in $\sigma$ can share at most one letter at the end of the first $\tau$-match and the start of the second $\tau$-match.

Examples:

- 1234 and 1324 are not minimal overlapping
- 132, 1243, and $12 \cdots (n-2)n(n-1)$ are minimal overlapping.

We applied the Reciprocity Method by Jones and Remmel to obtain descent type analogue of this result.
Reciprocity Method
The $n^{\text{th}}$ elementary symmetric function $e_n$ is defined by

$$E(t) = \sum_{n \geq 0} e_n t^n = \prod_i (1 + x_i t).$$

The $n^{\text{th}}$ homogeneous symmetric function $h_n$ is defined by

$$H(t) = \sum_{n \geq 0} h_n t^n = \prod_i \frac{1}{1 - x_i t} = \frac{1}{E(-t)}.$$

The $n^{\text{th}}$ power symmetric function $p_n$ is defined by

$$p_n = \sum_i x_i^n.$$

If $\lambda = (\lambda_1, \ldots, \lambda_{\ell})$ is a partition, then $h_\lambda = \prod_{i=1}^{\ell} h_{\lambda_i}$, $e_\lambda = \prod_{i=1}^{\ell} e_{\lambda_i}$, and $p_\lambda = \prod_{i=1}^{\ell} p_{\lambda_i}$.
Brick tabloid of shape \((n)\) and type \(\lambda\): an \(1 \times n\) rectangle chopped into "bricks" of lengths found in the partition \(\lambda\).

For example, if \(n = 8\) and \(\lambda = (1^2, 3^2)\) then the six brick tabloids of shape \((n)\) and type \(\lambda\) are:

\[
\begin{array}{cccc}
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\end{array}
\]
Let $\mathcal{B}_{\lambda,(n)}$ be the set of all of brick tabloids of shape $(n)$ and type $\lambda$, and let $B_{\lambda,(n)} = |\mathcal{B}_{\lambda,(n)}|$. 

Eğecioğlu and Remmel proved that

$$h_n = \sum_{\lambda \vdash n} (-1)^{n-\ell(\lambda)} B_{\lambda,(n)} e_\lambda.$$
Let
\[ NM_{\tau,n}(x, y) = \sum_{\sigma \in NM_n(\tau)} x^{LR_{\min}(\sigma)} y^{1 + desc(\sigma)} \]

Once again, $NM_n(\tau)$ is the set of permutations of length $n$ with no $\tau$-matches.

We consider the generating functions of the form
\[ NM_\tau(t, x, y) = \sum_{n \geq 0} \frac{t^n}{n!} NM_{\tau,n}(x, y) \]
Jones and Remmel showed that if $\tau$ starts with 1 then

$$NM_\tau(t, x, y) = \left( \frac{1}{U_\tau(t, y)} \right)^x$$

where

$$U_\tau(t, y) = 1 + \sum_{n \geq 1} U_{\tau, n}(y) \frac{t^n}{n!}$$

Thus,

$$U_\tau(t, y) = \frac{1}{1 + \sum_{n \geq 1} NM_{\tau, n}(1, y) \frac{t^n}{n!}}$$

We will apply the homomorphism method to give a combinatorial interpretation of the right hand side of the above equation.
Homomorphism Method

We define a homomorphism $\theta$ on the set of symmetric functions $\Lambda$ as follows

$$
\theta(e_n) = \frac{(-1)^n}{n!} NM_{\tau,n}(1, y)
$$

Then,

$$
\theta(E(-t)) = \sum_{n \geq 0} NM_{\tau,n}(1, y) \frac{t^n}{n!} = \frac{1}{U_{\tau}(t, y)}
$$

Hence,

$$
U_{\tau}(t, y) = \frac{1}{\theta(E(-t))} = \theta(H(t)) \implies n! \theta(h_n) = U_{\tau,n}(y)
$$
Therefore, to compute the generating function $NM_\tau(t, x, y)$ we first need to compute $n!\theta(h_n)$ to obtain the polynomials $U_{\tau,n}(y)$

Then

$$U_\tau(t, y) = 1 + \sum_{n \geq 1} U_{\tau,n}(y) \frac{t^n}{n!}$$

and ultimately,

$$NM_\tau(t, x, y) = \left(\frac{1}{U_\tau(t, y)}\right)^x$$
Jones and Remmel showed that one can compute $n!\theta(h_n)$ in several cases where $\text{des}(\tau) = 1$ and $\tau$ starts with 1.

Using the recursion formula from Eğecioğlu and Remmel

$$h_n = \sum_{\lambda \vdash n} (-1)^{n - \ell(\lambda)} B_{\lambda, (n)} e_\lambda.$$  

Jones and Remmel showed that

$$n!\theta(h_n) = \sum_{\lambda \vdash n} (-1)^{\ell(\lambda)} \sum_{B \in B_{\lambda, (n)}} \binom{n}{b_1, \ldots, b_{\ell(\lambda)}} \prod_{i=1}^{\ell(\lambda)} N M_{\tau, b_i}(1, y)$$

where $B = (b_1, \ldots, b_{\ell(\lambda)}) \in B_{\lambda, (n)}$
Previous Results from Jones & Remmel

\[ n! \theta(h_n) = \sum_{\lambda \vdash n} (-1)^{\ell(\lambda)} \sum_{B \in B_{\lambda,(n)}} \binom{n}{b_1, \ldots, b_{\ell(\lambda)}} \prod_{i=1}^{\ell(\lambda)} NM_{\tau,b_i}(1, y) \]

where \( B = (b_1, \ldots, b_{\ell(\lambda)}) \in B_{\lambda,(n)} \)

Combinatorial interpretation for the right hand side of the previous equation in terms of brick tabloids:

- **Fix a brick** \( B = (b_1, \ldots, b_{\ell(\lambda)}) \)
- **Fill the cells of** \( B \) **with a permutation** \( \sigma = \sigma_1 \ldots \sigma_n \) **such that within each brick** \( b_i \), the entries in the cells reduce to permutation in \( \mathcal{N}M_{b_i}(\tau) \)
- **Label each descents of** \( \sigma \) **that occurs within a brick by** \( y \)
- **Label** \(-y\) **at the end of each bricks** \( b_i \)
Combinatorial interpretation for the right hand side of the previous equation in terms of brick tabloids:

- Fix a brick $B = (b_1, \ldots, b_{\ell(\lambda)})$
- Fill the cells of $B$ with a permutation $\sigma = \sigma_1 \ldots \sigma_n$ such that within each brick $b_i$, the entries in the cells reduce to permutation in $\mathcal{N}\mathcal{M}_{b_i}(\tau)$
- Label each descents of $\sigma$ that occurs within a brick by $y$
- Label $-y$ at the end of each bricks $b_i$

For example, when $n = 10$ and $\tau = 1324$, a possible object created this way is

\[
\begin{array}{ccc}
-y & y & y & -y \\
4 & 5 & 10 & 9 & 1 & 2 & 8 & 6 & 7 & -y \\
1 & 2 & 3 & 5 & 1 & 2 & 4 & 3 & 2 & 1
\end{array}
\]
Next, Jones and Remmel defined an involution on $B_{\lambda,(n)}$ as follows

Scan the cells from left to right, looking for the first cell $c$ such that either

- $c$ is labeled with a $y$ or
- $c$ is at the end of brick $b_i$, $\sigma_c > \sigma_{c+1}$, and there is no $\tau$-match that lies entirely within the cells of $b_i$ and $b_{i+1}$

The involution is sign-reversing, weight-preserving so the fixed-points of this involution give us $n!\theta(h_n) = U_{\tau,n}(y)$.
Lemma

Let $O = (B, \sigma)$ be a fixed point of the involution. Then $O$ satisfies the following conditions:

- The elements in each brick of $O$ are increasing.
- The first element of each brick of $O$ form an increasing sequence, reading from left to right.
- For any two consecutive bricks $b_i$ and $b_{i+1}$ in $O$, either
  - (a.) There is an increase between $b_i$ and $b_{i+1}$, or
  - (b.) There is a decrease between $b_i$ and $b_{i+1}$ but there is a $\tau$-match that lies entirely within the cells of $b_i$ and $b_{i+1}$

For example, when $\tau = 1423$, a typical fixed point of this involution is

\[
\begin{array}{cccccc}
-y & \ 1 & 2 & 9 & -y & \ 3 & 8 & -y & \ 4 & 5 & 6 & -y & \ 7 & 10 & -y
\end{array}
\]
Jones and Remmel showed that for certain $\tau$, the fixed points of the involutions satisfied certain natural recurrences which allowed them to show that the $U_{\tau,n}(y)$ satisfied simple recurrence relations.

**Theorem**

For $\tau = 1324$, $U_{\tau,1}(y) = -y$, and for $n \geq 2$,

$$U_{\tau,n}(y) = (1 - y)U_{\tau,n-1}(y) + \sum_{k=2}^{\lfloor n/2 \rfloor} (-y)^{k-1} C_{k-1} U_{\tau,n-2k+1}(y)$$

where $C_k = \frac{1}{k+1} \binom{2k}{k}$ is the $k$-th Catalan number.
Reciprocity Method for Family of Patterns

If $\Gamma$ is a set of permutations, then we let $\mathcal{N}\mathcal{M}_n(\Gamma)$ be the set of permutations $\sigma \in S_n$ which have no consecutive occurrences of any of the permutations in $\Gamma$.

We can extend Jones and Remmel’s method to study the generating function

$$M(\Gamma)(x, y, t) = \sum_{n \geq 0} M_{\Gamma, n}(x, y) \frac{t^n}{n!} = \sum_{n \geq 0} \frac{t^n}{n!} \sum_{\sigma \in \mathcal{N}\mathcal{M}_n(\Gamma)} x^{LRmin(\sigma)} y^{1 + \text{des}(\sigma)}$$

where $\Gamma$ is a set of permutations such that for all $\tau \in \Gamma$, $\tau$ starts with 1 and $\text{des}(\tau) \leq 1$.

This was my talk from two years ago at this conference!
Now if $\tau$ is a permutation that starts with 1 and has more than one descents then the current involution no longer applies. Here is a counter example for $\tau = 14253$.

Fix: put more conditions on our involution to ensure that we are breaking and combining the bricks at the same position.
Suppose that $\alpha = \alpha_1 \ldots \alpha_j$ and $\beta = \beta_1 \ldots \beta_j$ are permutations in $S_j$ such that $\alpha_1 = \beta_1 = 1$, $\alpha_j = \beta_j$, $\text{des}(\alpha) = \text{des}(\beta)$, and $\alpha$ and $\beta$ have the minimal overlapping property. Then

$$\sum_{n \geq 0} \frac{t^n}{n!} \sum_{\sigma \in \mathcal{N}\mathcal{M}_n(\alpha)} x^{\text{des}(\sigma)} = \sum_{n \geq 0} \frac{t^n}{n!} \sum_{\sigma \in \mathcal{N}\mathcal{M}_n(\beta)} x^{\text{des}(\sigma)}.$$ 

Thus $\alpha$ and $\beta$ are des-c-Wilf equivalent.

If in addition, $\text{inv}(\alpha) = \text{inv}(\beta)$, then

$$\sum_{n \geq 0} \frac{t^n}{[n]_q!} \sum_{\sigma \in \mathcal{N}\mathcal{M}_n(\alpha)} x^{\text{des}(\sigma)} q^{\text{inv}(\sigma)} = \sum_{n \geq 0} \frac{t^n}{[n]_q!} \sum_{\sigma \in \mathcal{N}\mathcal{M}_n(\beta)} x^{\text{des}(\sigma)} q^{\text{inv}(\sigma)}.$$ 

Thus $\alpha$ and $\beta$ are $(\text{des}, \text{inv})$-c-Wilf equivalent.
Examples

Let us consider permutations of length 5 and start with 1.

- There is no minimal overlapping permutation that starts with 1 and ends with 5.
- There is only one minimal overlapping permutation that starts with 1 and ends with 4, namely 12354.
- Also, since the permutation starts with 1, in order to be minimal overlapping, it cannot end with a rise. There are now 10 cases.
Examples

<table>
<thead>
<tr>
<th>$\sigma$</th>
<th>des($\sigma$)</th>
<th>inv($\sigma$)</th>
<th>Minimal overlapping?</th>
</tr>
</thead>
<tbody>
<tr>
<td>12453</td>
<td>1</td>
<td>1</td>
<td>yes</td>
</tr>
<tr>
<td>12543</td>
<td>2</td>
<td>3</td>
<td>yes</td>
</tr>
<tr>
<td>14253</td>
<td>2</td>
<td>3</td>
<td>no</td>
</tr>
<tr>
<td>15243</td>
<td>2</td>
<td>4</td>
<td>no</td>
</tr>
<tr>
<td>13452</td>
<td>1</td>
<td>3</td>
<td>yes</td>
</tr>
<tr>
<td>13542</td>
<td>2</td>
<td>4</td>
<td>yes</td>
</tr>
<tr>
<td>14352</td>
<td>2</td>
<td>4</td>
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</tr>
<tr>
<td>14532</td>
<td>2</td>
<td>5</td>
<td>yes</td>
</tr>
<tr>
<td>15342</td>
<td>2</td>
<td>5</td>
<td>yes</td>
</tr>
<tr>
<td>15432</td>
<td>3</td>
<td>6</td>
<td>yes</td>
</tr>
</tbody>
</table>

des-c-Wilf equivalence classes: $\{13542, 14352, 14532, 15342\}$. 
### Examples

<table>
<thead>
<tr>
<th>$\sigma$</th>
<th>des($\sigma$)</th>
<th>inv($\sigma$)</th>
<th>Is minimal overlapping?</th>
</tr>
</thead>
<tbody>
<tr>
<td>12453</td>
<td>1</td>
<td>1</td>
<td>yes</td>
</tr>
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<td>3</td>
<td>yes</td>
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<td>yes</td>
</tr>
<tr>
<td>15432</td>
<td>3</td>
<td>6</td>
<td>yes</td>
</tr>
</tbody>
</table>

(des, inv)-c-Wilf equivalence class: {13542, 14352} and {14532, 15342}. 
The size of the \((\text{des}, \text{inv})\)-c-Wilf equivalence classes can get arbitrarily large as \(n\) goes to infinity. Observe the following:

- If \(\sigma = 1 \sigma_2 \cdots \sigma_{n-1} 2 \in S_n\) then it is automatically minimal overlapping.
- For every three consecutive elements \(x, x+1, x+2\), the sequences \(t_1(x) = (x+1)(x+2)x\) and \(t_2(x) = (x+2)x(x+1)\) each have one descent and two inversions.

So we construct permutations of the form

\[1 \ t_i(3) \ t_i(6) \ t_i(9) \cdots \ t_i(3n) \ 2 \quad \text{for } i = 1, 2\]

to see that the \((\text{des}, \text{inv})\)-c-Wilf equivalence class has size at least \(2^n\).
Thank You.